STABILITY OF QUASI-ONEDIMENSIONAL MAGNETOHYDRODYNAMIC FLOWS

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Stability of onedimensional magnetohydrodynamic flows in channels of variable cross-section with particular attention given to the growth of small perturbation waves, was investigated in [1 and 2]. In [1] the wave amplification factor was determined under the assumption of homogeneity of the flow (its parameters independent of x), while in [2] amplification factors for a nonhomogeneous flow were found with the help of the magnetic field induction equation.

Investigation of stability should include a statement of the boundary value problem for linearised nonsteady equations and a determination of natural frequencies λ.

Below we consider a problem of stability of a quasi-onedimensional magnetohydrodynamic flow at small magnetic Reynold's numbers with respect to short wave oscillations (unlike [2], we assume that electric and magnetic fields are any specified functions of x) Parameters of the flow which is investigated for stability, are functions of the longitudinal coordinate x, while parameters of the perturbed state are functions of x and t. Boundary conditions at the front and back end of the magnetohydrodynamic channel are obtained from the requirement of continuous transition of the mode of flow into a pure gasdynamic flow perturbed only by the waves expanding from the boundaries of the magnetohydrodynamic channel. (We assume that no perturbations arrive at these boundaries from the outside). Under these assumptions, we investigate three different modes of flow: supersonic, subsonic and a flow in which sonic transition takes place within the shock wave.

Stability of steady flows with a continuous transition through the sonic barrier is also investigated in the linear approximation. As we know, continuous sonic transition occurs at the singular points of the system of equations describing a steady flow [3].

Below we show that continuous transition from supersonic to subsonic flow is stable, if it occurs at a node, and unstable if it occurs at a saddle point. The opposite is true in the case of a transition from subsonic to supersonic flow.

1. Let us consider a flow of a nonviscous fluid without heat conduction, in a plane channel of variable cross-section $y(x)$, in the presence of an external magnetic field $\mathbf{B} = (0, 0 - B)$. Electrical conductivity of this fluid is $\sigma = \sigma(p, \rho)$, upper and lower walls of the channel are conducting and exhibit a potential difference of 2Φ within $0 \leqslant x \leqslant L$, while outside this segment they are nonconducting. Magnetic field intensity B and electric potential Φ are, on the segment $0 \leqslant x \leqslant L$, any specified functions of x, while outside this segment they are identically equal to zero

$$B = \begin{cases} 0 & (x < 0, \quad x > L), \\ B(x) & (0 \leqslant x \leqslant L), \end{cases} \qquad \varphi = \begin{cases} 0 & (x < 0, \quad x > L) \\ \varphi(x) & (0 \leqslant x \leqslant L) \end{cases} \tag{1.1}$$

Direction of velocity vector of the flow coincides with the positive direction of the x-axis.

Continuity, motion and energy equations of an unsteady onedimensional flow at small Reynold's numbers, have the form

$$y\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u y)=0$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}=-\frac{\partial p}{\partial x}+\sigma B\left(\frac{\varphi}{y}-uB\right) \quad (1.2)$$

$$y\frac{\partial p}{\partial t}+uy\frac{\partial p}{\partial x}+\varkappa p\frac{\partial}{\partial x}(uy)=(\varkappa-1)\sigma y\left(\frac{\varphi}{y}-uB\right)^2$$

where u is velocity, ρ is the density, p is the pressure of fluid and $\varkappa$ is the ratio of specific heats. Equation of state for a perfect gas was utilised in deriving the above equations.

In order to investigate the stability of the flow defined by steady state equations corresponding to (1.2), we shall consider a linearised system

$$\frac{\partial \rho}{\partial t}+\frac{(Uy)'}{y}\rho+U\frac{\partial \rho}{\partial x}+\frac{(Ry)'}{y}u+R\frac{\partial u}{\partial x}=0$$

$$(UU'-B\alpha\sigma_\rho)\rho+R\frac{\partial u}{\partial t}+(RU'+\sigma B^2)u+RU\frac{\partial u}{\partial x}-B\alpha\sigma_p p+\frac{\partial p}{\partial x}=0$$

$$-(\varkappa-1)\alpha^2\sigma_\rho\rho+\left(P'+\varkappa P\frac{y'}{y}+2(\varkappa-1)\sigma B\alpha\right)u+\varkappa P\frac{\partial u}{\partial x}+\frac{\partial p}{\partial t}+$$

$$+p\left(\frac{\varkappa(Uy)'}{y}-(\varkappa-1)\alpha^2\sigma_p\right)+U\frac{\partial p}{\partial x}=0 \quad (1.3)$$

where $R(x)$, $U(x)$ and $P(x)$ are the density, velocity and pressure respectively of the steady flow; $\rho(x, t)$, $u(x, t)$ and $p(x, t)$ are perturbations of the corresponding magnitudes and are assumed small; σ_p and σ_ρ are the corresponding partial derivatives of electrical conductivity σ and $\alpha = \varphi/y - UB$. Derivatives with respect to x are denoted by a prime. Since the system (1.2) will not be used again, we shall retain its notation for perturbations.

2. We shall seek the solution of (1.3) in the form

$$\rho(x, t)=u_1(x)e^{\lambda t}, \qquad u(x, t)=u_2(x)e^{\lambda t}, \qquad p(x, t)=u_3(x)e^{\lambda t} \quad (2.1)$$

since the growth of functions $\rho(x, t), \ldots$, when $t\to\infty$, is governed [4] by the extreme right-hand side eigenvalue λ.

Putting (2.1) into (1.3) we obtain a system of ordinary linear differential equations. Solution of this system can be found by expanding it into an asymptotic series in $\varepsilon = 1/\lambda$, provided that $|\lambda| > N$ where $N > 0$ is sufficiently large [5].

Let us write a general expression for this system

$$B_{ij}u_j'+(\lambda A_{ij}+C_{ij})u_j=0 \qquad (i, j=1, 2, 3) \quad (2.2)$$

where repeated index denotes summation. Writing its solution in the form of a series

$$u_j=(u_{j0}+\varepsilon u_{j1}+\varepsilon^2 u_{j2}+\ldots)\exp\int h\,dx$$

and inserting it into (2.2), we obtain a sequence of systems of linear algebraic equations defining u_{j0}, u_{j1} etc.

$$(B_{ij}\varepsilon h+A_{ij})\,u_{j0}=0 \quad (2.3)$$

$$(B_{ij}\varepsilon h+A_{ij})u_{j1}=-(C_{ij}u_{j0}+B_{ij}u'_{j0}) \quad (2.4)$$

Condition

$$|B_{ij}\varepsilon h+A_{ij}|=0$$

is necessary for a nontrivial solution of (2.3) to exist, and it yields the values of $(h\varepsilon)^{(k)}$ $(k = 1, 2, 3)$ at which the determinant of (2.3) becomes zero. A solution $u_{j0}^{(k)} = u_{j0*}^{(k)} f^{(k)}(x)$ corresponds to each value of $(h\varepsilon)^{(k)}$, and $f^{(k)}(x)$ is an arbitrary function which can be found from the condition of compatibility of (2.4), with the accuracy of up to the constant multiplier

$$v_i^{(k)} B_{ij} u_{j0*}^{(k)} f^{(k)\prime} + (C_{ij} u_{j0*}^{(k)} + B_{ij} u_{j0*}^{(k)\prime})\, v_i^{(k)} f^{(k)} \equiv G(f) = 0$$

where $v_i^{(k)}$ are solutions of (2.3) with a transposed matrix.

Solution of (2.4) gives

$$u_{j1}^{(k)} = u_{j1*}^{(k)} f^{(k)}(x) + u_{j0*}^{(k)} f_1^{(k)}(x)$$

Function $f_1^{(k)}(x)$ is found from the condition

$$G(f_1^{(k)}) + [C_{ij} u_{j1*}^{(k)} f^{(k)} + B_{ij} (u_{j1*}^{(k)} f^{(k)})']\, v_i^{(k)} = 0$$

of compatibility of a system defining u_{j2}. This gives $f_1^{(k)} = cf^{(k)} + f_{1*}^{(k)}$, where c is an arbitrary constant which can be assumed zero.

Subsequent approximations $u_{j2}, u_{j3}, \ldots$ are found in an analogous manner. Functions $f^{(k)}(x)$ and $f_{1*}^{(k)}(x)$ are determined with accuracy of up to an arbitrary constant multiplier c_k which is common to all these functions and which can be found from the boundary conditions.

General solution of (2.2) can be written as

$$u_j = c_k u_j^{(k)} = c_k f^{(k)}(x)\,[u_{j0*}^{(k)} (1 + \varepsilon (f_{1*}^{(k)} / f^{(k)}) + \ldots) + \varepsilon u_{j1*}^{(k)} + \ldots] \exp \int h^{(k)}\, dx \qquad (2.5)$$

For (1.3), we have

$$h^{(1)} = -\lambda / U, \quad h^{(i)} = -\lambda / (U \pm a) \qquad (a = \sqrt{\varkappa P / R})$$

$$u_{10*}^{(k)} = 1, \quad u_{20*}^{(1)} = u_{30*}^{(1)} = 0, \quad u_{20*}^{(i)} = \pm a / R, \quad u_{30*}^{(i)} = a^2$$

$$u_{11*}^{(k)} = 0, \quad u_{21*}^{(1)} = U (Ryf^{(1)})^{-1} (Uyf^{(1)})', \quad u_{31*}^{(1)} = U (UU' - B\alpha\sigma_\rho)$$

$$u_{21*}^{(i)} = (U \pm a)(Ryf^{(i)})^{-1} [(U \pm a) yf^{(i)}]'$$

$$\begin{aligned} u_{31*}^{(i)} = (U \pm a) \{ [a (a \pm U)]' + a (a \pm U) f^{(i)\prime} / f^{(i)} + U' (U \pm a) \pm \\ \pm aUy' / y \pm aB^2 \sigma / R - B\alpha (\sigma_\rho + a^2 \sigma_p) \} \pm aRu_{21*}^{(i)} \end{aligned} \qquad (2.6)$$

$$f^{(1)}(x) = \frac{R(x)}{R(0)} \exp \int_0^x K_1\, dx \qquad (k = 1, 2, 3, i = 2, 3)$$

$$f^{(i)}(x) = \left(\frac{a^3(0)\, R(x)}{a^3(x)\, R(0)} \right)^{1/2} \exp \int_0^x K_i\, dx$$

where functions under the integral sign have the form

$$K_1 = -\frac{(\varkappa - 1)\, \alpha^2 \sigma_\rho}{a^2 U}, \qquad K_i = \frac{\pm aB + (\varkappa - 1)\, \alpha}{2a^2 (U \pm a)}\, \alpha (a^2 \sigma_p + \sigma_\rho) -$$

$$- \frac{\sigma B^2}{2R (U \pm a)} \mp \frac{(2\varkappa - 1)\, \sigma B\alpha}{2aR (U \pm a)} - \frac{(\varkappa + 1)\, U'}{2 (U \pm a)} + \frac{y' (\varkappa U \pm a)}{2y (U \pm a)}$$

Magnitudes equal to $-1/(h\,\varepsilon)^{(k)}$ $(k = 1, 2, 3)$ represent the velocities of propagation of small perturbations. In the case of a supersonic flow, all three perturbation waves move downstream, while in the case of a subsonic flow, two waves move downstream and the remaining one moves upstream with velocity of $(U - a)$. Zero approximations u_{j0*} correspond to the analogous gasdynamic solution.

Expressions for $f^{(k)}(x)$ yield wave amplification factors. For a supersonic flow they are

$$\frac{f^{(1)}(x)}{f^{(1)}(0)} = \frac{R(x)}{R_0} \exp \int_0^x K_1\, dx, \qquad \frac{f^{(i)}(x)}{f^{(i)}(0)} = \left(\frac{a_0^3 R(x)}{R_0 a^3(x)} \right)^{1/2} \exp \int_0^x K_i\, dx \qquad (i = 2, 3)$$

For a subsonic flow the above form is retained for the waves propagated with velocities U and $U + a$, while for the wave whose velocity is $(U - a)$, it becomes

$$\frac{f^{(3)}(x)}{f^{(3)}(L)} = \left(\frac{a_L{}^3 R(x)}{R_L a^3(x)}\right)^{1/2} \exp \int_L^x K_3 dx$$

Subscripts 0 and L refer to parameters in the cross-sections $x = 0$ and $x = L$ respectively.

3. In order to determine the coefficients c_1, c_2 and c_3, let us write the boundary conditions for three possible modes of flow: supersonic, subsonic and mixed, the latter exhibiting a sonic transition within the shock wave. We shall assume that the waves arriving at the boundaries of a magnetohydrodynamic channel, will be of zero amplitude. This will be true, e.g. for an open working cycle of the channel.

Amplitudes of waves arriving at the boundaries $(x = 0, x = L)$ may be different from zero if the waves spreading from the magnetohydrodynamic channel undergo a reflection within a gasdynamic part of the flow. These waves can be taken into account when formulating the boundary conditions. Such a reflection however will not take place if walls of the channel smoothly diverge on approach to a large size receiver and towards an exit into the atmosphere.

Let us denote the solution by u_j^- when $x < 0$ and by u_j^+ when $x > L$. Assuming that the solution of (1.3) is continuous at $x = 0$ and $x = L$ i.e. that a magnetohydrodynamic flow is continuously transformed into a gasdynamic flow, we obtain three types of boundary conditions.

Supersonic flow. In this case, all three small perturbation waves move downstream Assumption that external perturbations are absent implies that $u_j = 0$ when $x = 0$, hence c_k is given by

$$c_k u_j^{(k)} = 0 \qquad (j, k = 1, 2, 3)$$

It is easy to show that the determinant $|u_j^{(k)}|$ of this system is, for large λ, different from zero, hence c_k has only a trivial solution. This means that eigenfunctions are not formed and that any initial perturbation will be removed beyond the boundaries of the channel in a finite period of time.

Subsonic flow. Here one wave moves into the region $x < 0$ hence $u_j^- = c_3^- u_j^{-(3)}$, while at $x = L$, we have two waves moving into the region $x > L$ which gives

$$u_j^+ = c^+{}_1 u_j^{+(1)} + c_2^+ u_j^{+(2)}$$

Condition of continuity of gasdynamic parameters on the boundaries of a channel yields

$$c_k u_j^{(k)} = c_3^- u_j^{-(3)} \quad (x = 0), \quad c_k u_j^{(k)} = c_1^+ u_j^{+(1)} + c_2^+ u_j^{+(2)} \quad (x = L)$$
$$(j, k = 1, 2, 3) \tag{3.1}$$

A necessary condition for the eigenfunction to exist is, that the $D(\lambda)$-determinant of the system (3.1) which defines c_1, C_2, c_3, c_1^+, c_2^+ and c_3^- is equal to zero,

$$D(\lambda) = -2\varepsilon^2 (u_{31\bullet}^{(3)})_0 (aRu_{21\bullet}^{(1)} - u_{31\bullet}^{(1)})_L f^{(1)}(L) \exp\left(-\lambda \int_0^L \frac{dx}{U}\right) +$$
$$+ \varepsilon^2 (aRu_{21\bullet}^{(3)} + u_{31\bullet}^{(3)})_0 (aRu_{21\bullet}^{(2)} - u_{31\bullet}^{(2)})_L f^{(2)}(L) \exp\left(-\lambda \int_0^L \frac{dx}{U+a}\right) +$$
$$+ 4a_0{}^2 a_L{}^2 f^{(3)}(L) \exp\left(-\lambda \int_0^L \frac{dx}{U-a}\right) = 0 \tag{3.2}$$

The above equation is used to determine the natural frequencies λ.

In the derivation of (3.2) we have assumed that the condition (1.1) is fulfilled, that the derivative $\gamma'(x)$ has a discontinuity and that $\gamma' = 0$ as $x \to -0$ and $x \to +L$. If γ' is continuous, then Expressions (2.6) for $u_{1\bullet}$ contain no terms with γ'

It should be noted that when u_j is taken in zero approximation only, then $D(\lambda) \neq 0$ and consequently all $c_k = 0$. This follows from the fact that at large λ, coefficients of reflection of waves from the ends of the channel are of order ε. Reflections are caused by discontinuities in φ, B, y, y' and U' occurring on the boundaries of the channel and intensities of the cause and effect are directly proportional to each other.

First two terms of (3.2) contain $f^{(1)}(L)$ and $f^{(2)}(L)$, which represent exponential functions of (2.6). Therefore, if

$$\int_0^L K_1\,dx > \int_0^L K_2\,dx$$

then the second term is small compared with the first and can be neglected. If, on the other hand

$$\int_0^L K_2\,dx > \int_0^L K_1$$

then we can neglect the first term. Thus, (3.2) can be split into two equations, each of them valid under the conditions given above, and each reducing to

$$1/\lambda^2 = \xi_l \exp \lambda\eta_l \qquad (l = 1, 2) \tag{3.3}$$

$$\xi_1 = \frac{2a_0^2 a_L^2 f^{(3)}(L)}{(u_{31*}^{(3)})_0\,(aRu_{21*}^{(1)} - u_{31*}^{(1)})_L\, f^{(1)}(L)}, \qquad \eta_1 = \int_0^L \frac{a\,dx}{U(a-U)} > 0$$

$$\xi_2 = \frac{4a_0^2 a_L^2 f^{(3)}(L)}{(aRu_{21*}^{(3)} + u_{31*}^{(3)})_0\,(u_{31*}^{(2)} - aRu_{21*}^{(2)})_L\, f^{(2)}(L)}, \qquad \eta_2 = \int_0^L \frac{2a\,dx}{a^2 - U^2} > 0 \tag{3.4}$$

Onset of instability is associated with the presence of such natural frequencies λ, for which Re $\lambda = \lambda_r > 0$. For (3.3) this requirement for roots $|\lambda| > N$ is fulfilled, when

$$|\xi_l| < 1/N^2 \qquad (l = 1, 2) \tag{3.5}$$

Indeed, roots of (3.3) represent, in the complex plane λ, a set of isolated points distributed along a continuous curve

$$\text{Im}\,\lambda = \lambda_i(\lambda_r) = \pm\left(\frac{1}{|\xi|}\exp(-\eta\lambda_r) - \lambda_r^2\right)^{1/2}$$

When (3.5) holds, then the curve $\lambda_i(\lambda_r)$ intersects the imaginary axis in the λ-plane, outside a circle of radius N, where expansion into a series in $1/\lambda$ is valid, and we can always find a root of (3.3) with a positive real part near the point of intersection, since at small λ_r we have $\lambda_i = (^1/_2\pi \pm 2m\pi)/\eta$, where m is an integer.

From (3.4) it follows that ξ_1 and ξ_2 are small, if corresponding Expressions

$$\exp\int_0^L (K_1 - K_3)\,dx, \qquad \exp\int_0^L (K_2 - K_3)\,dx$$

are large. This is possible when $K_1 - K_3 > 0$ and $K_2 - K_3 > 0$.

If we assume that electric conductivity is a power function of temperature $\sigma = T^n$, then reduction of (3.4) to dimensionless form yields

$$\xi = \xi(\Phi, M, Y, n, \Delta, \varkappa), \quad \Phi = \frac{\varphi}{yaB}, \quad M = \frac{U}{a}, \quad Y = \frac{y'RU}{y\sigma B^2}, \quad \Delta = \frac{\sigma B^2 L}{UR}$$

Fig. 1 shows the curves for various values of parameters n and Y ($n = 0.5$, $Y = -1, 0, 1$, $\varkappa = {}^5/_3$) dividing the (Φ, M)-plane into two regions Ω_1 and Ω_2. In Ω_1 condition $K_1 > K_2$ holds, while in Ω_2 the condition $K_2 > K_1$. If parameters of the flow are in the region Ω_1 over the whole or on the greater part of the segment $0 \leqslant x \leqslant L$, then Eq. (3.3) with $l = 1$ holds, while if the parameters are in Ω_2, then (3.3) with $l = 2$ holds. This means that the growth or decay of perturbations in a magnetohydrodynamic flow in Ω_1 is governed by the reflection of small perturbation waves propagating with velocities U and $U - a$, while in Ω_2 the behavior of perturbations is controlled by reflection of the waves propagating with velocities $U + a$ and $U - a$. In near vicinity of the line of discontinuity reflection of all three waves takes place and Eq. (3.2) must then be used.

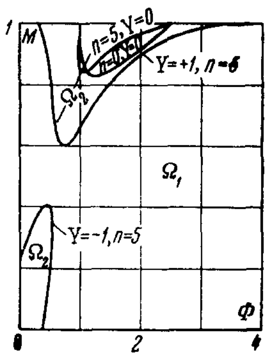

Fig. 1

Figs. 2*a* and *b* give curves for various values of parameters n and Y. These curves separate region Q of possible instability of the flow where ξ_1 is small ($K_1 - K_3 > 0$), from the region where ξ_1 is large ($K_1 - K_3 < 0$). If parameters of the flow are within the Q-regions at all $x \in [0, L]$, then such flow may be unstable. It will however be stable if these parameters fall outside the Q-regions.

Region of instability becomes larger with increasing n when $\gamma' = 0\,(Y = 0)$ and this, basically, leads to a mode in which energy is supplied to the gas ($\Phi > M$). (Fig. 2*a* gives the curves for parameter values $Y = 0$, $n = 0, 5, 10$, $\varkappa = {}^5/_3$). In the diverging channels ($\gamma' > 0$) this corresponds to positive values of Y and region of instability becomes larger, while in the converging channels ($\gamma' < 0$, $Y < 0$) region of instability diminishes and shifts to the right towards larger values of Φ (see Fig 2*b*).

(Fig. 2*b* gives the curves defining a region Q of possible instability for the following parameter values $Y = 1, 0, -1$, $n = 5$, $\varkappa = {}^5/_3$).

Points at which these curves intersect the line $M = 1$, are singular points of a stationary system of equations.

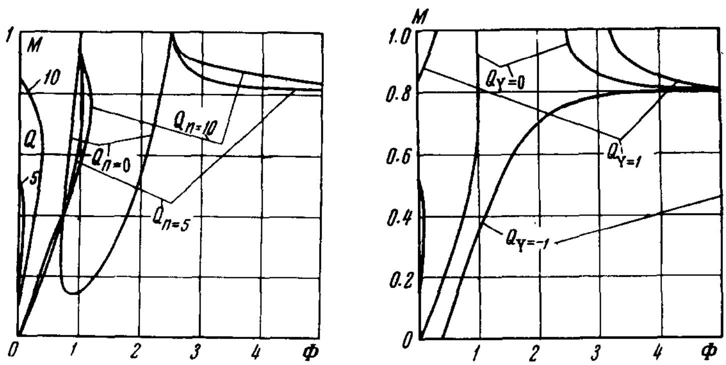

Fig. 2*a*, *b*

We have found that Eq. (3.3) with $l = 2$ does not produce other regions of instability, since the condition sign $(K_2 - K_3) =$ sign $(K_1 - K_3)$ holds in the regions Ω_2 (see Fig. 1).

Instability which we have obtained here is, in the physical sense, analogous to the global instability discussed in [6].

Flow with a shock wave. We have a steady flow in a channel and we assume that a shock wave appears at the cross-section $x = 0$ irrespective of whether the channel is, at $x < 0$, gasdynamic or magnetohydrodynamic. In an unsteady flow, position of the shock wave is given by $x = \vartheta e^{\lambda t}$ (ϑ is an arbitrary constant), since by previous assumption the factor $e^{\lambda t}$ defines the dependence of our solution on time, and the displacement velocity of the shock wave is given by $dx/dt = \lambda \vartheta e^{\lambda t}$ where $\lambda\vartheta$ is small, while λ is, as before, large.

Linearising the conditions for gasdynamic parameters on the shock wave (shock wave is gasdynamic, since the conductivity is assumed finite) and substituting solutions (2.5) for $x = 0$ into the resulting equations we obtain, neglecting terms of order $1/\lambda$,

$$c_k \left(U_0 u_1^{(k)} + R_0 u_2^{(k)}\right) - \vartheta\lambda \left(R_0 - R_0^-\right) = 0$$

$$c_k \left(U_0^2 u_1^{(k)} + 2U_0 R_0 u_2^{(k)} + u_3^{(k)}\right) = 0$$

$$c_k \left(-\frac{\varkappa}{\varkappa - 1}\frac{P_0}{R_0^2} u_1^{(k)} + U_0 u_2^{(k)} + \frac{\varkappa}{\varkappa - 1}\frac{1}{R_0} u_3^{(k)}\right) - \vartheta\lambda \left(U_0 - U_0^-\right) = 0$$

Here the superscript$^-$ and the subscript$_0$ denote the parameters in front of the shock wave, while the subscript$_0$ alone, the parameters behind the shock wave. When $x = L$, second condition of (3.1) holds

$$c_k u_j^{(k)} = c_1^- u_j^{-(1)} + c_2^- u_j^{-(2)} \qquad (j,\ k = 1,\ 2,\ 3)$$

Resulting system of six homogeneous equations with six unknowns c_1, c_2, c_3, c_1^-, c_2^- and ϑ has a nontrivial solution, when the determinant

$$D(\lambda) = -2(M_0+1)^2(M_0+\varkappa_1)a_L{}^2 f^{(3)}(L)\exp\left(-\lambda\int_0^L\frac{dx}{U-a}\right) +$$

$$+\varepsilon(aRu_{21*}^{(2)} - u_{31*}^{(2)})_L(M_0-1)^2(M_0-\varkappa_1)f^{(2)}(L)\exp\left(-\lambda\int_0^L\frac{dx}{U+a}\right) + \tag{3.6}$$

$$+2\varepsilon(aRu_{21*}^{(1)} - u_{31*}^{(1)})_L(M_0{}^2-1)\left(\varkappa M_0+\frac{1}{M_0}\right)f^{(1)}(L)\exp\left(-\lambda\int_0^L\frac{dx}{U}\right) = 0$$

Here $\varkappa_1 = (3-\varkappa)/2(\varkappa-1)$. In the course of derivation of (3.6) we have utilised known relations on the shock wave.

Using similar arguments we find, that the onset of instability in presence of a shock wave is subject to the condition

$$|\zeta_l| < 1/N \qquad (l = 1,\ 2)$$

which differs from (3.5) in values of ζ_l and in the fact that the denominator contains N and not N^2.

From the physical point of view it means, that the coefficient of reflection from the shock wave is of the order of unity, while that of reflection from the end $x = L$ is, as before, of the order ε. Therefore onset of instability is more possible in presence of a shock wave. Expressions for ζ_l have the form

$$\zeta_1 = \frac{a_L{}^2 M_0(M_0+1)(M_0+\varkappa_1)f^{(3)}(L)}{(aRu_{21*}^{(1)} - u_{31*}^{(1)})_L(M_0-1)(\varkappa M_0{}^2+1)f^{(1)}(L)}$$

$$\zeta_2 = \frac{2a_L{}^2(M_0+1)^2(M_0+\varkappa_1)f^{(3)}(L)}{(aRu_{21*}^{(2)} - u_{31*}^{(2)})_L(M_0-1)^2(M_0-\varkappa_1)f^{(2)}(L)} \tag{3.7}$$

from which it follows that regions of possible instability are the same as in the case of subsonic flow, since their denominators contain the same exponential terms.

From (3.7) it follows that strong shock waves produce strong reflections which enhance the instability, while in the case of weak shock waves ($M_0 \to 1$) the reflection may become so weak, that a reflection from the cross-section where magnitudes y', B, φ, and U' exhibit a discontinuity, may have a decisive influence, i.e. a case discussed previously.

4. Let us consider the behavior of small perturbations near the points of continuous sonic transition in a steady flow.

If any one of characteristic velocities becomes zero at some point x^* of the considered segment of x-axis, then the theorem which states that the asymptotic behavior in time of the solution of the linearised system is governed by the factor $e^{\lambda t}$, is no longer valid. Therefore we shall investigate the distribution of perturbations over a finite duration of time and space. We shall limit ourselves to such perturbations, which can be represented by a Fourier integral in large values of λ only. This is true for wave packets and for individual short impulses of any form, when the contribution of small values of λ to the Fourier integral can be neglected.

Then a solution for an unsteady perturbation equation can be written in zero approximation, as

$$u_j = \sum_{k=1}^{3} u_{j0*}^{(k)} f^{(k)}(x)\int F^{(k)}(\lambda)\exp\left(\lambda t - \lambda\int\frac{dx}{c^{(k)}(x)}\right)d\lambda, \qquad c^{(k)}(x) = -\frac{1}{(h\varepsilon)^{(k)}} \tag{4.1}$$

where $F^{(k)}(\lambda)$ is a Fourier representation of a perturbation propagating with velocity $c^{(k)}(x)$.

Let us now consider a term, corresponding to some $c^{(k)}$ in (4.1). We have, along the characteristic $dx/dt = c^{(k)}(x)$,

$$\lambda t - \lambda \int \frac{dx}{c^{(k)}(x)} = \text{const}$$

From this it follows, that a perturbation described by this term propagates along the characteristics belonging to a corresponding family, and growth of perturbations is defined by $f^{(k)}(x)$.

Characteristic perturbation wavelength varies as $c^{(k)}(x)$. Short impulses expand without changing their form (amplitude and scale change along the x-axis).

Let us assume that, at some point $x = x^*$, a continuous transition occurs, from a supersonic, to a subsonic steady flow. Then the perturbations whose velocity is equal to $U - a$, approach the point x^* from both sides, but achieve it only when $t \to \infty$. At the same time, as we said before, amplitude of the wave will either tend to infinity or remain bounded together with $f^{(3)}(x)$.

In the first case when the magnitude of perturbation increases without bounds, we shall assume the flow to be unstable.

If, on the other hand, a transition from subsonic to supersonic flow occurs at $x = x^*$, then perturbations corresponding to $u_j^{(3)}$ originate in the vicinity of the transition point. The ratio $f^{(3)}(x_1)/f^{(3)}(x)$ gives the amplification of these perturbations over the time t, and x denotes a point at which this perturbation was situated at $t = 0$, which arrived at the point x_1 at the instant t.

With fixed x_1 and $t \to \infty$, we have $x \to x^*$. Thus, if initial perturbations are bounded everywhere, then perturbations will, at all points, grow without bounds if $f^{(3)}(x) \to 0$ as $x \to x^*$, and decay if $f^{(3)}(x) \to \infty$ as $x \to x^*$. In the first case we have an unstable flow.

We shall show that if sonic transition takes place in a saddle point for the steady state equations then $f^{(3)}(x) \to \infty$ as $x \to x^*$, while if the transition occurs at the nodal point for the steady state equations, then $f^{(3)}(x) \to 0$ as $x \to x^*$.

The following relations hold [3] at the singular point $x = x^*$, $U = a$

$$\frac{y'}{y} = \frac{\sigma B^2}{aR} \varkappa (\Phi - 1)\left(\frac{\varkappa - 1}{\varkappa} \Phi - 1\right), \qquad U' = \frac{2UM'}{\varkappa + 1} + \frac{\varkappa - 1}{\varkappa + 1} \frac{\sigma B^2}{R} \Phi (\Phi - 1)$$

$$\Phi = \frac{\varphi}{yBa}, \quad M' = -\frac{\gamma_1}{2} \pm \frac{\gamma_1}{2}\left(1 - 4\frac{\gamma_2}{\gamma_1^2}\right)^{1/2} \tag{4.2}$$

$$\gamma_1 = \frac{\sigma B^2}{aR} \frac{\varkappa(\varkappa + 1)}{2}\left(1 - \frac{\varkappa - 1}{\varkappa}\Phi\right)\left(1 - \frac{\varkappa}{\varkappa + 1}\Phi\right), \qquad \gamma_2 = \gamma_2(y, \varphi, B, y', y'', \varphi', B', \varkappa, x^*)$$

Substituting (4.2) into the expression for $f^{(3)}$ and under the integral sign in (2.6), we obtain

$$\frac{1}{2(x - x^*)M'}\left[-\gamma_1 \mp \gamma_1\left(1 - 4\frac{\gamma_2}{\gamma_1^2}\right)^{1/2}\right] \tag{4.3}$$

Here similarly to [3] we assume, that σ – const (we note that when $\sigma = \sigma(p, \rho)$, Expression (4.3) retains its form).

In [3] we have also shown that: (1) When $\gamma_2 > 0$ and $1 - 4\gamma_2/\gamma_1^2 > 0$, then the singular point is a node and also if $\gamma_1 > 0$, then $M' < 0$ and we have a transition from supersonic to subsonic flow, while if $\gamma_1 < 0$, then $M' > 0$ and we have a transition in the opposite direction; (2) When $\gamma_2 < 0$, singular point is a saddle point and from it the flow passes from subsonic to supersonic in one direction ($M' > 0$), while in the other direction where $M' < 0$, the flow passes from supersonic to subsonic.

In the case of a nodal point, the sign of the expression for $f^{(3)}$ under the integral sign is defined by the sign of $(-\gamma_1 M')$; when $\gamma_1 > 0$ and $M' < 0$ or when $\gamma_1 < 0$, $M' > 0$, the integrand is positive and $f^{(3)} \to 0$ as $x \to x^*$. In the case of a saddle point, the sign of the integrand is always negative and $f^{(3)} \to \infty$ as $x \to x^*$.

In conclusion we shall note that the present investigation was concerned with linearised equations. If nonlinear terms were taken into account in (1.3), then the behavior of perturbation amplitudes near a saddle point might differ from that obtained for the linear approximation.

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BIBLIOGRAPHY

1. Sutton, G.W., Witalis, E.A., Linearized analysis of MHD generator flow stability. Internat. sympos. maghetohydrodynam. electr. power generation, Vol. 2, Paris, 1964.
2. Artiushkov, E.V. and Morozov, A.I., On the longitudinal stability of onedimensional flows of a conducting gas. Teplofizika vysokikh temperatur, Vol. 2, No. 4, 1964.
3. Slobodkina, F.A., Qualitative investigation of equations of quasi-onedimensional magnetohydrodynamic flow in a channel. PMTF, No. 3, 1966.
4. Brushlinskii, K.V., On the growth of solution of a mixed problem in case of incomplete eigenfunctions. Izv. Akad. Nauk SSSR, Ser. Mat., Vol. 23, No. 6, 1959.
5. Heading J. An Introduction to Phase-Integral Methods. Wiley, 1962.
6. Kulikovskii, A.G., On the stability of homogeneous states. PMM Vol. 30, No. 1, 1966.

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